

Exact Chiral Symmetry for Domain Wall Fermions with Finite L_s^*

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We show how the standard domain wall action can be simply modified to allow arbitrarily exact chiral symmetry at finite fifth dimensional extent L_s . We note that the method can be used for both quenched and dynamical calculations. We test the method using smooth and thermalized gauge field configurations. We also make comparisons of the performance (cost) of the domain wall operator for spectroscopy compared to other methods such as the overlap-Dirac operator and find both methods are comparable in cost.

1. The problem with small eigenvalues

Recently a great deal of theoretical progress has been made in the construction of lattice regularizations of fermions with good chiral properties [1,2]. For use in practical numerical simulations, though, approximations to these formulations are necessary. In the formulation using domain wall fermions (DWF) [1,3], the extent of an auxiliary fifth dimension has to be kept finite in numerical simulations while chiral symmetry holds strictly only in the limit of infinite fifth dimension. The violations of chiral symmetry are expected to be suppressed exponentially in the extent of the fifth dimension [3,4], but in practice the coefficient in the exponent can be quite small [5,6] and the suppression correspondingly slow.

In the case of overlap fermions there is no such problem in principle. However, there is a problem of practicality: how to deal efficiently with $\epsilon(H) = H/\sqrt{H^2}$, where H is some auxiliary Hermitian lattice Dirac operator for large negative mass, but free of doublers. Most commonly, the Hermitian Wilson-Dirac operator H_w is used.

Any good numerical approximation to $\epsilon(H)$ must retain the property $\epsilon(H)^2 = 1$ which is crucial for the Ginsparg-Wilson relation

$$\{\gamma_5, D_{ov}(0)\} = 2D_{ov}(0)\gamma_5 D_{ov}(0) \quad (1)$$

to hold and exact chiral symmetry for the overlap

Dirac operator [7]

$$D_{ov}(m) = \frac{1}{2} [1 + m + (1 - m)\gamma_5 \epsilon(H)] \quad (2)$$

at $m = 0$ to be preserved. A good approximation of $\epsilon(x)$ is difficult to achieve for small x .

Unfortunately, at least in quenched simulations with the Wilson gauge action, $H_w(M) = \gamma_5 D_w(-M)$, with $D_w(M)$ the Wilson-Dirac operator, has a non-vanishing density of zero eigenvalues, $\rho(0; M)$, for any gauge coupling β [8]. $\rho(0; M)$ decreases quite rapidly with increasing β , roughly as $\rho(0; M)/\sigma^{3/2} \sim e^{-e^\beta}$ [8], but at currently used couplings a considerable number of modes with small eigenvalues exists.

In all numerical methods to implement overlap fermions one can enforce accuracy of the approximation to $\epsilon(H_w)$ by projecting out the lowest few H_w eigenvectors and adding their correct contribution exactly (see Ref. [9]): with $H_w v_i = \lambda_i v_i$,

$$\begin{aligned} \epsilon(H_w) &= \sum_{i=1}^n |v_i\rangle \epsilon(\lambda_i) \langle v_i| + \mathcal{P}_\perp^{(n)} \text{App}[\epsilon(H_w)] \mathcal{P}_\perp^{(n)}, \\ \mathcal{P}_\perp^{(n)} &= \mathbf{1} - \sum_{i=1}^n |v_i\rangle \langle v_i|. \end{aligned} \quad (3)$$

The number n of projected eigenvectors is chosen such that the approximation $\text{App}[\epsilon(H_w)]$ to $\epsilon(H_w)$ is sufficiently accurate in the subspace spanned by $\mathcal{P}_\perp^{(n)}$.

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2. Implications for domain wall fermions

Integrating out the heavy fermions, one can show that the effective Dirac operator for the physical 4-d fermions from the domain wall fermion approach is [10,11]

$$D_{tov}(m) = \frac{1}{2} \left[1 + m + (1 - m) \gamma_5 \varepsilon_{L_s/2}(H_T) \right] \quad (4)$$

with $\varepsilon_N(x)$ the polar decomposition approximation to $\epsilon(x)$ [12]

$$\varepsilon_N(x) = \frac{(1+x)^{2N} - (1-x)^{2N}}{(1+x)^{2N} + (1-x)^{2N}} \approx \tanh(2Nx) \quad (5)$$

and

$$H_T(M) = H_w(M) \frac{1}{2 + \gamma_5 H_w(M)}. \quad (6)$$

Zero modes of $H_w(M)$ are also zero modes of $H_T(M)$!

Small eigenvalues of H_T induce chiral symmetry violations, unless L_s is very large — $\tanh(y)$ is within about 10^{-5} of 1 for $y \geq 6.1$. And the density of small eigenvalues of H_T is the same as the density of small eigenvalues of H_w !

Manifestations of the induced chiral symmetry breaking are, for example, violations of the Gellmann-Oakes-Renner relation

$$m\chi_\pi \equiv m \langle \sum_x \pi(x) \pi(0) \rangle = \langle \bar{\psi} \psi \rangle, \quad (7)$$

finding a non-zero pion mass for massless quarks, and absence of exact zero modes in topologically non-trivial gauge field backgrounds. This latter is illustrated in Fig. 1 where we show the spectral flow of the hermitian domain wall operator $H_{DW}(0; M) = \gamma_5 \mathcal{J} D_{DW}^{(5)}(0; M)$, with \mathcal{J} the inversion operator of the fifth direction, the hermitian Wilson operator $H_w(M)$ and the hermitian overlap $H_{ov}(0; M) = \gamma_5 D_{ov}(0; M)$ on a single instanton configuration. Right after a Wilson eigenvalue crosses zero the overlap operator has an exact zero mode while the lowest eigenvalue of the domain wall operator is clearly non-zero even for $L_s = 32$ for M considerably after the crossing.

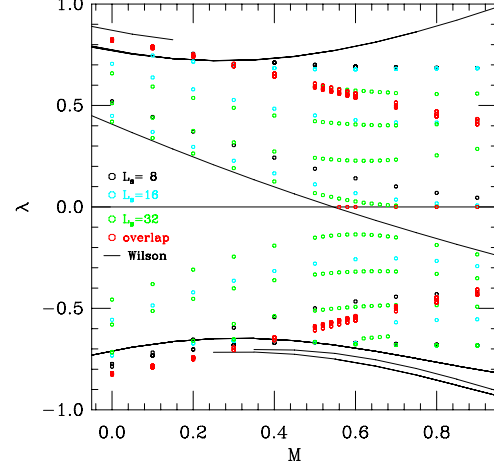


Figure 1. Eigenvalues of the hermitian domain wall, Wilson and overlap operators as function of the negative Wilson mass value M on a single instanton configuration.

3. Exact chiral symmetry at finite L_s

With finite lattice spacing in the 5th direction, a_5 , the domain wall fermion (DWF) action is

$$S_{DW} = -\bar{\Psi} D_{DW}^{(5)} \Psi. \quad (8)$$

The kernel of the 5-d operator is, including the boundary conditions used to introduce an explicit quark mass m ,

$$D_{DW}^{(5)} = \begin{pmatrix} D_+ - \hat{A}P_- & -P_- & \cdots & 0 & (m - \hat{A})P_+ \\ -P_+ & D_+ & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_+ & -P_- \\ mP_- & 0 & \cdots & -P_+ & D_+ \end{pmatrix} \quad (9)$$

where $P_\pm = (1 \pm \gamma_5)/2$ and $D_+ = a_5 D_w(-M) + 1$. $\hat{A}(m)$ is the new ingredient, acting on the light fermions, chosen to insure chiral symmetry. We integrate out the $L_s - 1$ extra fermion fields and

the pseudo-fermions to obtain [11]

$$\begin{aligned}
& D_{tov}(m; H_T) \\
&= \left\{ \mathcal{P}^{-1} \left[D_{DW}^{(5)}(1) \right]^{-1} D_{DW}^{(5)}(m) \mathcal{P} \right\}_{11} \\
&= \frac{1}{2} \left[1 + m + (1 - m) \gamma_5 \varepsilon_{L_s/2}(a_5 H_T) \right] \\
&\quad + \text{term with } \hat{A}(m) .
\end{aligned} \tag{10}$$

Here $\varepsilon_N(x)$ is the polar decomposition approximation to $\epsilon(x)$ introduced in (5), and \mathcal{P} is such that $(\mathcal{P}^{-1}\Psi)_1 = q$ is the light fermion field,

$$\mathcal{P}_{jk} = \begin{cases} P_- \delta_{j,k} + P_+ \delta_{j+1,k} & \text{for } j < L_s \\ P_- \delta_{L_s,k} + P_+ \delta_{1,k} & \text{for } j = L_s . \end{cases} \tag{11}$$

H_T , given in (6), is the auxiliary Hamiltonian associated with the transfer matrix T in the 5^{th} direction,

$$T \equiv \frac{1 - a_5 H_T}{1 + a_5 H_T} . \tag{12}$$

With $H_T v_i = \lambda_i v_i$, $T v_i = T_i v_i$ we set,

$$\hat{A}(m) = (1 - m) \gamma_5 (H_w P_- - 1) \sum_{i=1}^n g_i |v_i\rangle \langle v_i| , \tag{13}$$

$$g_i = \frac{1}{2} \left[- \left(T_i^{-L_s} - 1 \right) + \left(T_i^{-L_s} + 1 \right) \epsilon(\lambda_i) \right] . \tag{14}$$

This “projects” low-lying eigenvalues of H_T from $\varepsilon_{L_s/2}(a_5 H_T)$ and gives the correct $\epsilon(\lambda_i)$. Then $D_{tov}(m; H_T)$ becomes $D_{ov}(m; H_T)$,

$$D_{ov}(m; H_T) = \frac{1}{2} [1 + m + (1 - m) \gamma_5 \epsilon(H_T)] , \tag{15}$$

the overlap operator with auxiliary Hamiltonian H_T .

With projection of the 5 lowest eigenvectors of H_T we now find the expected zero-modes of the massless projected domain wall operator $\sqrt{(D_{DW}^{(5)}(0))^\dagger D_{DW}^{(5)}(0)}$ for the same instanton configuration as in Fig. 1 already for moderate L_s , as can be seen in Fig. 2.

As already mentioned, the Gellmann-Oakes-Renner relation (7) is a sensitive test of chiral symmetry. For overlap fermions

$$m \langle b | (\gamma_5 \tilde{D}_{ov}^{-1}(m))^2 | b \rangle = \langle b | \tilde{D}_{ov}^{-1}(m) | b \rangle \tag{16}$$

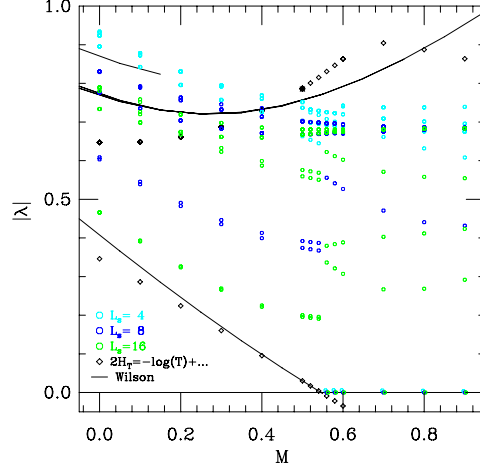


Figure 2. Eigenvalues of the domain wall operator with projection as function of the negative Wilson mass value M on a single instanton configuration. Also show is the Wilson operator spectral flow and the eigenvalues of $2H_T(M)$.

holds for any chiral vector $\gamma_5 |b\rangle = \pm |b\rangle$, and averaging over several chiral Gaussian vectors $|b\rangle$ this becomes a stochastic estimate for the Gellmann-Oakes-Renner relation (7). In (16) $\tilde{D}_{ov}^{-1}(m)$ is the external fermion propagator with the contact term subtracted and multiplicatively normalized [9], which avoids $\mathcal{O}(a)$ effects [13],

$$\begin{aligned}
(1 - m) \tilde{D}_{ov}^{-1}(m) &= [D_{ov}^{-1}(m) - 1] \\
&= \left[\left\{ \mathcal{P}^{-1} \left[D_{DW}^{(5)}(m) \right]^{-1} D_{DW}^{(5)}(1) \mathcal{P} \right\}_{11} - 1 \right] .
\end{aligned} \tag{17}$$

We note that for standard DWF without projection the subtraction and multiplicative normalization is done automatically, *i.e.*, with $\bar{q} = (\bar{\Psi} \mathcal{J} \mathcal{P})_1$ where $\mathcal{J}_{ij} = \delta_{i, L_s+1-j}$ is the inversion operator of the fifth direction,

$$\begin{aligned}
\langle q \bar{q} \rangle &= \left\{ \mathcal{P}^{-1} [D_{DW}^{(5)}(m)]^{-1} \mathcal{J} \mathcal{P} \right\}_{11} \\
&= \frac{1}{1 - m} [D_{tov}^{-1}(m) - 1] .
\end{aligned} \tag{18}$$

Finite L_s and no projection lead to strong violations of the Gellmann-Oakes-Renner relation (7). With projection of a sufficient number of low eigenvectors the GMOR relation holds down to much lower quark masses, as seen in Fig. 3.

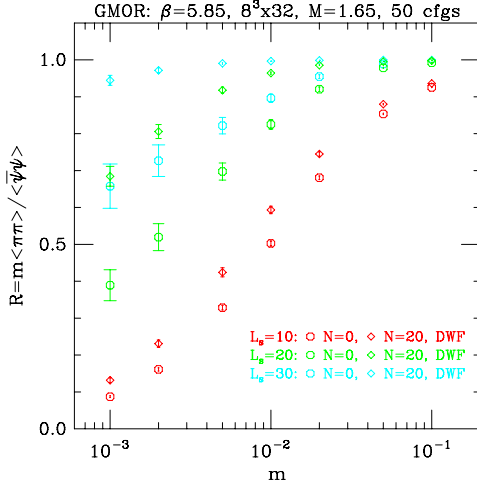


Figure 3. The ratio $R = m_{\chi\pi} / \langle \bar{\psi}\psi \rangle$ for DWF without and with projection of the lowest 20 eigenvectors.

4. Comparing Efficiencies

Having different possibilities for chiral fermion formulations on the lattice we can compare their efficiencies in numerical simulations. For this we consider a spectroscopy calculation for a topologically trivial gauge configuration. The cost is measured in the number of times D_w , common to all approaches, has to be applied on a 4-d vector. This is shown in Fig. 4.

For a single quark mass preconditioned DWF, with projection to assure minimal induced chiral symmetry breaking (see Fig. 3) at negligible cost, appears most efficient. However, in a quenched spectroscopy calculation with overlap fermions one can use a multi-shift inverter to compute for all quark masses simultaneously, compensating for the factor 3-4 in increased cost. Which regularization to prefer depends thus on the details of the numerical simulation attempted. If DWF are chosen then the projection method described here and in [11] should definitely be used to ensure good chiral properties at moderate L_s .

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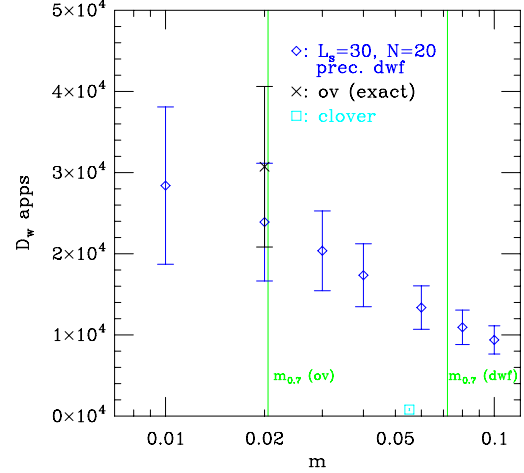


Figure 4. The cost of various implementations of chiral fermions as measured by the number of applications of D_w on a 4-d vector.

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